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THE OCCURRENCE OF A DOMINANT SPOTTING MUTATION IN THE HOUSE MOUSE

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The gene *W* for dominant spotting in mice was identified by one of us (1915) as having a lethal effect when present in a homozygous condition and as being independent in inheritance from the gene for piebald spotting (*sp*).

Investigations by de Aberle and others (1927) showed that the lethal effect of the gene *W* was due to anemia and that while some *WW* individuals might remain alive for ten days after birth, the majority died earlier—many of them in utero.

Recently in a strain of silvered black self mice of the formula *wwBBslslSS* an exceptional male appeared (No. 638). He had a number of small irregular white spots or streaks on his dorsal surface and a white streak on the mid-ventral line. The colored areas of his coat were also markedly paler than the typical coat color of the black silvered stock from which he arose. Various crosses of this mouse and of its descendants have given clear evidence that either a mutation of *w* to *W* or to a new allele *W'* has occurred.

Experimental.—Male 638, the exceptional spotted animal, was crossed with his self black silvered sister 639 and gave a total of 23 young of which 13 were "exceptional" spotted silvers like himself, 7 were ordinary black silvered self mice and three died before the color could be recorded.

Beginning with the offspring of the 639 × 638 mating and continuing through several generations a number of matings were made between "exceptional" spotted mice. These matings gave a total of 98 young. Of these, 52 resembled the parents, being "exceptional" spotted, 36 were self silvered and ten were *black-eyed whites*. Of these, one died at some unidentified date during the first three weeks, *one lived to be an adult mouse* and the others died at 17, 20, 20, 22, 22, 23, 25 and 26 days, respectively. In external appearance they did not differ from the black-eyed whites previously described, but their length of survival was strikingly different from that of other strains previously recorded.

We may for convenience designate the new gene for dominant spotting as W'. The formulae of the "exceptional" spotted mice would then be *W'wSSBBslsl*. Certain of these mice were then crossed with two black self females from strain *C*57. These animals were *wwSSBBS^{L}S^{L}* in formula. As expected there were two color classes in F_1. One class was black self of the fomula *wwSSBBslsl*. The other consisted of animals all of which had some ventral white and which usually had one or more small white patches on the dorsal surface. These were *W'wSSBBS^{L}sl* in genetic formula. Their ground color was not clear deep black, but was distinctly more brown and more gray than would be expected in *BB* animals.

Since the *W'wBBslsl* "exceptional" spotted mice were distinctly modified in ground color as compared with the ordinary black silvered mice, it would appear that *W'* and two "doses" of *sl* produce this change. In the *S^{L}sl* mice one would ordinarily expect that dominance of *S^{L}* would be complete as far as superficial appearance was concerned. Apparently, however, the presence of *W'* changes this expected relationship between *S^{L}* and *sl* and the ground color is definitely modified.

Through the kindness of Dr. Law of the Bussey Institution, animals heterozygous for the gene *W* were obtained. These mice were crossed with black silvered *ww* mice of the same stock as that in which the *W'* mutation had occurred. A total of 24 young was produced. Of these 17 were self and 7 spotted. Little, if any, difference between the ground color of the self *wwS^{L}sl* and the spotted *WwS^{L}sl* mice could be observed. If there is any interaction between *W* and *sl* it is very definitely less than that between *W'* and *sl*.

Further crosses between the mice derived from *C*57 black $\times$ *W^{i}w* mice were next made. Self mice from this cross, presumably *wwS^{L}sl*, were crossed *inter se* and gave 17 selfs (none of which, however, were visibly silvered). The expectation was 3 self non-silvered to 1 self silvered. The significance of the absence of silvered mice in this mating is doubtful. The *W'wS^{L}sl* mice crossed *inter se* gave a total of 17 "modified" black spotted mice like the parents, 4 self and 6 black-eyed whites of which 1 died before 30 days, *one is alive at 35 days and four grew to be adult mice.*

Attempts at breeding the adult black-eyed whites have not been very successful. One male out of two and one female out of three are all that have been found to be fertile. There are also signs that the fertility is temporary and lasts only for a very limited time.

One litter was obtained from black-eyed whites crossed *inter se*. It consisted of three black-eyed whites of which one died at 15 days, one at 19 days and one at between 35 and 40 days. One litter from a *C*57 black female *ww* was obtained by a homozygous black-eyed white male *W'W'*. It consisted of five mice all "modified" black and spotted as would be expected if they were *W'w* in formula. One outcross litter of ten young was

obtained from a *Ww* female by a *W'W'* male. There were three black-eyed whites *W'W* which died at 19, 20 and 22 days, respectively, and seven modified black spotted probably *W'w* in formula. *The one fertile black-eyed white male thus sired a total of 18 young, all of which showed that they had obtained the W' gene in the gamete derived from him.*

There remain a large number of tests both physiological and genetic which should be carried out. We shall be glad to send mice with the gene *W'* to anyone caring to do such work.

Discussion.—From even the preliminary results, it seems probable that the lethal action of the *W* gene is a manifestation of that gene itself rather than a different gene closely linked to *W*, as has been suggested by Kobozieff (1934) and others.

The gene *W'* prolongs the life of homozygotes but still has a delayed lethal or sub-lethal action resulting in complete or partial sterility. This would seem to indicate a gradation of lethal effect more difficult to reconcile with the conception of a recessive lethal gene than with a new possible allele of *W* itself. It would be necessary to assume two simultaneous mutations in nearby loci to account for the appearance of dominant spotting and lethal action in animals with gene *W'* if Kobozieff's theory is correct.

Conclusion.—We may, therefore, conclude that a new dominant spotting mutation has occurred and that *W'* probably represents a new allele of *W* definitely more viable in a homozygous condition than was the original *W* gene.

[1] de Aberle, S. B., *Amer. Jour. Anat.*, **40**, 219–249 (1927).

[2] de Aberle, S. B., Hoskins, W. M., and Bodansky, M., *Jour. Biol. Chem.*, **72**, 643–648 (1927).

[3] Kobozieff, N., *Compt. Rend. Acad. Sci.*, **198**, 617–619 (1934).

SUCCESSION OF FOSSIL FLORAS IN PATAGONIA

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The present note is devoted to the more recent discoveries of fossil plants in Patagonia and makes no attempt to review what is known of fossil plants in that region from strata older than Upper Cretaceous. The theme has to do with the Age of Flowering Plants.

Although fossil floras, made up largely of diocotyledonous leaves, have been known for many years, especially from the region on both sides of the Straits of Magellan, there has been great uncertainty regarding their age, and a tendency to confuse them with younger floras, such as those known since 1891 from the Concepcion-Arauco coalfields of Chile, and those

described in more recent years by the present writer from various localities in Rio Negro, Chubut and Santa Cruz territories of Argentina.

Most of the old discoveries and several of the newer ones have the plants unassociated with either invertebrate or mammalian fossils, so that there has been no opportunity for any check on their ages, since the materials for a standard paleobotanical section were wanting, the stratigraphy was wholly of a reconnaissance nature, and there were no comparative paleobotanical sections nearer than North America.

Prior to 1925 the only records of Tertiary plants in print from southern South America, outside the Concepcion-Arauco district of Chile, were those from the Magellanean region. The botanical facies of the latter was rather similar throughout and the dominant plant forms were evergreen and other beeches. All of the leaves were small and tended to be coriaceous forms, and the climatic implications were distinctly temperate. There is no evidence that these assemblages represent a single geological horizon; in fact, it is rather clear that they do not. But they most decidedly indicate a readily recognizable facies and a climatic phase, which the present evidence shows to be rather sharply distinguishable from what preceded and what succeeded them in that region.

In one of the more elaborate accounts of this flora, that by Dusén, it was segregated into what he called an older Fagus zone and a younger Araucaria zone, and the respective ages were considered to be Oligocene and Aquitanian. There is the gravest doubt that, even if there is an Araucaria zone, it is as young as the Aquitanian stage. All that can be affirmed is that, in terms of the accepted geological sequence, the Fagus and Araucaria zones are older than the Patagonian stage, with its abundant marine faunas, which Ortmann considered as of Burdigalian age.

Dusén also described a very interesting Tertiary flora from Seymour Island off the east coast of Graham Land. He considered this to be older than his Fagus zone of Patagonia. Wilckens correlated the associated and poorly preserved mollusca with what he called the Patagonian molasse, and in this he was followed by Andersson (1906), although the presence of Zeuglodon, if correctly identified, stamps it as not younger than upper Eocene (Jackson stage of our Gulf Coastal Plain section, or slightly earlier). My opinion is that the Seymour Island plants are of approximately the same age as those on the South American mainland, since several have been recognized in new collections from this general horizon in Patagonia.

A collection from a locality on the Rio Turbio in Santa Cruz Territory sent to me recently by John O. Bower and Frank B. Notestein is interbedded with marine mollusca which the Texas Company paleontologists consider to be Oligocene, but whether this rests on Hatcher's section near Punta Arenas, or upon more convincing considerations I do not know.

For the purposes of the present contribution I will call the plants from

the so-called Fagus zone, the Araucaria zone and the Seymour Island Tertiary the Eogene flora. I have seen collections from a considerable number of new localities, the most interesting being from those discovered by George Gaylord Simpson, the leader of the Scarrett Patagonian expeditions, in Chubut Territory. These derive their main interest from the fact that they are the first representatives of the Eogene flora that have been found in association with mammal remains. These show that this flora is associated with mammals of the Notostylops, and possibly the Pyrotherium beds, which in Simpson's revised nomenclature of 1933 are the equivalent of the Casamayor, Musters and Deseado formations. This Eogene flora has also been found at several localities in the Lago Nahuel Huapi region, thus extending its range northward nearly to Latitude 40°S. from its outposts on Seymour Island.

In 1925 I described a considerable suite of fossil plants collected by Burton Clark at a locality which was christened Mirhoja in Chubut Territory, and which I considered to be of lower Miocene age, and correlated with the Concepcion-Arauco flora of Chile, and suggested that, in part at least, it was the same age as the so-called Santa Cruz beds, which have yielded such a marvelous fauna.

This lower Miocene flora has since been found as far south as the Rio Chalia in Santa Cruz Territory, and at several other localities, the most prolific being a locality on the Rio Pichileufu east of Lago Nahuel Huapi, the collections from which I owe to the enthusiasm of Señor Guinazu and the coöperation of the Geological Survey of Argentina. An account of this extensive flora is now in press. This flora is entirely distinct from the Eogene floras, not only in the genera and species represented, but in the environment indicated, and contains a large number of forms whose most similar existing species occur in lower latitudes in South America. It shows every evidence of containing large numbers of tropical and subtropical elements which here reached their southernmost limits, which must have been at a time that was unusually favorable for such a spread, and it extends about 10 degrees south of the most northerly known occurrences of the Eogene flora.

Within the past year I have received two collections from the Geological Survey of Argentina collected by A. Piatnitzkyi of that organization. One of these, from a locality known as Cerro Baguales in the Sehuen valley, Territory of Santa Cruz, is of late Upper Cretaceous age. It has been recently commented upon in *Science*, since its discovery has served to fix the age of the Cerro Guido flora described by Kurtz in 1899, the two being practically the only known Upper Cretaceous plant localities in the whole of South America. The second is from a locality known as Cerro Funes, near the boundary between Chubut and Santa Cruz territories. The latter is entirely distinct and younger than the former. Although relatively

small it consists entirely of broad-leafed dicotyledons, suggestive of a milder climate than that indicated by the Eogene flora, to some elements of which it appears to stand in an ancestral relationship, especially in its Fagaceous forms.

In existing floras the true beeches (Fagus) are confined to the northern hemisphere, and are represented in the South American, Australian and New Zealand regions by the genus Nothofagus. The question of the relation of Nothofagus to Fagus, and how and when it reached these isolated antipodean lands is an interesting one. In this Cerro Funes flora there is a species of Dryophyllum, the first so far as I know from the southern hemisphere. Dryophyllum is an extinct Cretaceous-Eocene genus, long known and common in the northern hemisphere, and one which I have considered as the ancestor from which the oaks, chestnuts and beeches took their origin. The beeches are the only one of these three stocks that has occupied these southern lands. There is present also at Cerro Funes a form which I have called Fagophyllum, since it appears to be intermediate between Dryophyllum and Fagus.

Both Fagus and Nothofagus are excessively common in the Eogene flora of southern South America as has already been mentioned, and they also appear as dominant elements in certain Tertiary floras in Australia and New Zealand, associated in the latter with an extinct genus known as Parafagus. In all these three regions the only member of the Fagaceae which has survived to modern times is the genus Nothofagus, with perhaps a dozen species, some of which are evergreen and some deciduous. Although so ancient a type Nothofagus shows no signs of having passed its zenith.

The other elements in this Cerro Funes flora also represent genera that are usually found in late Cretaceous and early Eocene floras of the northern hemisphere with one exception—a representative of the family Flacourtiaceae, which is essentially an austral alliance. Obviously the relationships and the source of this Cerro Funes flora was north of Patagonia, but whether northern in this case need be extended beyond the equatorial region is a debatable question.

The age of the Cerro Funes flora cannot be decisively settled in terms of the international time-scale. Regionally it is clearly younger than the late Cretaceous Cerro Baguales-Cerro Guido floras, and it is as definitely older than the Eogene Fagus floras. The collector Piatnitzky, calls the Cerro Funes flora Pehuenche in age, but he uses this term in a quite different sense from that in which it was used by Ameghino. Ameghino's Pehuenchense stage is of pre-Salamanca age and corresponds to what Simpson calls the Chubut formation, whereas the Pehuenche of Piatnitzky and some other contemporary authors is post-Salamanca and corresponds to what Simpson calls the Rio Chico formation.

The Rio Chico formation is considered to lie at the base of the Tertiary, but its lower boundary is not usually precisely located. In a sense it suggests a parallel with the so-called Laramie problem of North American geology, although this comparison may not hold in the light of more complete knowledge. I am inclined to think, however, that since all organisms, even fossil ones and both animal and vegetable, have ancestors and most of them have descendants, the effort to locate a definite objective boundary in a continuous series of continental deposits is a vain and foolish thing, whether the search lead to Patagonia, or to Montana, Dakota or Wyoming. The fates in the geological drama do not act with the celerity of the firing squad of a totalitarian state, even though given sufficient time and they will do a far more effective job. I am tentatively calling the Cerro Funes flora Paleocene (?).

To summarize, it appears that following the late Jurassic "cosmopolitan" floras, there continued through the Lower and Upper Cretaceous in the Patagonian region environmental conditions and land connections that did not place an embargo on the colonization of the area by a number of elements representing the Upper Cretaceous radiation of the Flowering plants, and that these conditions had not changed sufficiently by the dawn of the Tertiary to bring about any decided change in facies, even though the Cerro Funes plants are all specifically different from those found in the Upper Cretaceous Cerro Baguales flora.

At an undetermined time in the early Tertiary there was a change in environmental conditions the exact nature of which is unknown, but which had the effect of enabling what I have called the Eogene flora of small leafed and largely beech forest to attain a known extent from the Lago Nahuel Huapi region to the borders of Antarctica. This perhaps long-continued floristic episode was followed by a southward spread from equatorial America in the late Oligocene or lower Miocene of a wholly different flora which overlapped the Eogene flora by at least ten degrees of latitude.

Incidentally it may be noted that this lower Miocene flora was entirely replaced in late geological time in the whole region south of northern Argentina and east of the Andes by the modern and quite different plains and steppe flora of the present. In the modern narrow and wet strip west of the Andes and in certain intermontane valleys some survivors from the lower Miocene assemblage have maintained their existence and are mixed with many elements descended from the Eogene floras that once occupied the whole general region.

It seems certain, and this is the excuse for the present contribution, that the recent discoveries cited have established the following floral succession in Patagonia on a rather firm basis:

1. Cerro Baguales flora—late Cretaceous Salamanca stage.
2. Cerro Funes flora—Paleocene (?). Rio Chico formation.

3. Fagus-Araucaria zones—Eogene.
4. Lower Miocene flora—part equivalent of Santa Cruz.

However lacking in detail or in completeness, this sequence is certainly a great advance over our lack of knowledge of a few years ago, and seems destined to be a permanent contribution, whatever the precise correlation of these four floras turn out to be in terms of the world time-table.

SKIN TRANSPLANTS BETWEEN EMBRYOS OF DIFFERENT BREEDS OF FOWL

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These experiments were designed primarily to test the behavior of skin grafts of genetically different breeds of fowl. In this report we are to consider (1) the behavior of pigmented skin grafts on non-pigmented hosts and reciprocals and (2) the rôle of position in the body in determining feather tract (pteryla) pattern and form of the down feather.

Pieces of skin ectoderm are grafted between S. C. white leghorn embryos (white) and either F_1 embryos (black) of the cross Barred Rock ♀ × Rhode Island Red ♂[1] or Rhode Island Red embryos (red). At the time of operation both host and donor are incubated about 75 hours at a temperature of 37.7°C. The donor skin ectoderm is isolated in all cases from the dorso-lateral surface of the head anterior to the otocyst. The isolated piece of small dimensions—not exceeding 0.5 mm. wide × 1 mm. long—is freed of most, if not all, of the adhering mesoderm. In most cases it is vitally stained with neutral red which aids greatly in transferring and orientating it on the host. Through a window cut in the shell over the embryo the amnion is slit, the cut edges spread apart and an incision made by glass needles in the skin ectoderm at the base of the right wing or leg bud. The piece of donor skin ectoderm is now fitted into the incision, the window closed and the egg returned to the incubator. The embryos are removed and examined at about 15 days of age, at which time the down feathers are well developed. To prevent rotation during the operation and subsequent incubation the egg is kept on a nest of cotton in a watch-glass. Under these conditions development of the embryo ensues quite normally including the regeneration of the amnion.

Black Skin Grafted to White Hosts.—Of the sixteen living white host embryos obtained, all except four show irregularly shaped patches of black

down feathers. The position of these patches in general depends upon the site of the operation and to some extent upon the size of the original implant. When black skin ectoderm is grafted to the base of a right wing bud, black plumage develops on the entire surface (4 cases) or only the proximal portion (2 cases) of the wing. In other cases (4) as illustrated in figure 1, the black patch is even more extensive, covering not only the entire wing but adjacent portions of the breast, rump and the thigh of the

FIGURES 1 AND 2

1. White Leghorn embryo ($15^1/_2$-day) showing graft of black plumage.
2. White Leghorn embryo (15-day) showing graft from a Rhode Island Red embryo.

operated side. In one of these cases black feathers are missing from the terminal third of the wing and from the breast. When the implant is placed at the base of the leg bud the black plumage area developed covers the entire leg as in the normal and a portion of the rump (1 case) or is confined to the thigh and shank, except that the distal portion of the latter bears white feathers. One embryo was obtained in which a leg bud of a black donor was transplanted just posterior to the host's wing bud. In this case an extra leg covered with black plumage developed.

White Skin Grafted to Black Hosts.—Eighteen living black embryos which were hosts to a piece of white skin ectoderm have been recovered. In no single case did a patch of white plumage form. As a clue to this curious result attempts were made to follow the fate of the white skin by grafting either half or whole limb buds to black hosts. A similar result is obtained. The extra leg (4 embryos) or wing (3 embryos) which developed is covered with black plumage.

Red Skin Grafted to White Hosts and Reciprocals.—Two embryos have been examined. One of these shows no trace of the graft and the other shows an extensive patch of red plumage covering the entire wing except its terminal portion, adjacent portions of the breast, the back and even the thigh (figure 2). When white skin is grafted to red hosts none of the three embryos obtained shows any white plumage.

Differences in Behavior of Reciprocal Skin Grafts.—The results show that when skin ectoderm from pigmented breeds (F_1 black, or red) is grafted to unpigmented hosts (white), black or red plumage develops in accordance with the breed of the donor embryo. On the other hand, when unpigmented skin ectoderm is grafted to pigmented hosts patches of white feathers fail to appear. It is thus apparent that the fate of the graft is different in reciprocal graft-host combinations.

The nature of the reaction which takes place between the white skin graft and the black host is not known. One simple explanation is that the white skin grafted is invaded and replaced by the surrounding pigmented host epidermis. This may be expected on the basis of the findings of Rand and Pierce[2] for tadpoles and Saxon, Schmeckebier and Kelley[3] for guinea pigs, that autotransplants of white skin areas into pigmented areas become pigmented through invasion and replacement by the surrounding epidermis. An alternative hypothesis is that the white skin graft has persisted, the feathers thereon forming pigment through the action of chemical substances furnished by the host. As a result of the production (Durham[4]) and diffusion of such a substance as tyrosinase from the pigmented skin of the host, all potential white feathers transform into black ones. Evidence favoring this mode of action is brought out in the grafts of white limb buds into black hosts, where all feathers covering the extra limb developed are black. Such a limb has arisen undoubtedly from a white donor yet it is covered with black feathers. That the epidermal covering of the white limb bud mesoderm has been invaded and replaced by pigmented host epidermis seems improbable. Unfortunately no direct evidence for the persistence of the white skin epidermis could be obtained from a macroscopic examination of the limb.

Manner of Origin of the Black-Feathered Areas on White Hosts.—The area of black feathers may arise from the original implant of the skin ectoderm (1) solely by growth, i.e., without any effect on the surrounding white host

epidermis, (2) by growth accompanied by an outward invasion and replacement of the surrounding white epidermis with black epidermis such as Saxon, *et al.* (loc. cit.), have found for the guinea pig and (3) by growth accompanied by the production of a chemical substance which diffuses into the surrounding white epidermis bringing about a transformation of the white feathers into black ones. No final conclusion as to the exact mode of origin can be made at this time. That the black-feathered area arises entirely from the cells of the original implant regardless of whether or not any replacement of the white epidermis takes place seems probable. This is strongly supported by the fact that the feathers at the margin of the black areas are mosaics of black and white barbs longitudinally arranged. This may be seen through the transparent horny sheath. Half of the barbs of a single feather may be white and half black, one-third white and two-thirds black, etc. Such a barb composition indicates that the feather follicle producing them is composed of two kinds of epidermal cells derived in variable proportions from both donor and host (cf. Danforth and Foster[5]).

Assuming that the original implant is the source of the cells of the black-feathered area, the factors controlling the amount, paths and direction of growth in covering extensive regions of the body presents an interesting problem. That the amount of growth is many times that of the original piece is shown by the formation of a black area of large dimensions (2 × 3 cm.) from a tiny graft of skin ectoderm. That the amount is dependent upon inherent differences in growth rate of the skin ectoderm used seems probable. In general the path of growth extension is toward the tip of the limb when the implant is placed at the base of the limb rudiment; when placed in the wing position it also tends to spread posteriorly onto the back and thigh and anteriorly to a lesser extent. The forces responsible for these movements remain for future elucidation.

Feather Form and Tract Pattern Determined by Position in the Body.—A study of the distribution and form of the feathers in the black (or red) areas reveals their definite arrangement into tracts or *pterylae* and a size characteristic of the tract in which they occur. This is well brought out in cases where the black-feathered area covers an extensive portion of the body involving more than one feather tract. A single area in such cases may include the humeral and alar tracts on the wing, a portion of the spinal tract and the femoral and crural tracts of the leg. In each of these tracts the feathers have the characteristic arrangement and form found in corresponding tracts in control embryos and on the unoperated side (left) of the same embryo.[6] This is particularly striking on the wing where the down plumage shows much variation in arrangement and form, by the development of primaries, secondaries and coverts (major, median and minor). On the assumption (discussed above) that the black area develops from the original piece of skin ectoderm, such a variation in the form and distribu-

tion of the feathers in the different tracts clearly shows the importance of position in feather tract characterization at the stage of operation. This is further emphasized by the fact that the skin ectoderm implant came in all cases from the head where in the normal the down feathers are quite different in form (short) and arrangement (number of small tracts). It is evident, therefore, that the skin ectoderm at the time of grafting is dependent upon the mesodermal substratum or position in the body for the development of feather form and arrangement. On the contrary, the color of the feather is autonomous in its development, viz., in accordance with the genic constitution of the skin ectoderm cells themselves.

[1] In this cross the embryos have black plumage except for a patch of white plumage on top of the head in the males.

[2] Rand, H. W., and Pierce, M. E., *Jour. Exptl. Zoöl.*, **62**, 125–172 (1932).

[3] Saxon, J. A., Schmeckebier, M. M., and Kelley, R. W., *Biol. Bull.*, **71**, 453–461 (1936).

[4] Durham, F. M., *Proc. Roy. Soc. London*, **74**, 310–313 (1904).

[5] Danforth, C. H., and Foster, Frances, *Jour. Exptl. Zoöl.*, **52**, 443–470 (1929).

[6] For the embryology of the feather tracts in the chick, see Holmes, Anne, *Am. Jour. Anat.*, **56**, 513–537 (1935).

GENERAL PROJECTIVE DIFFERENTIAL GEOMETRY[1]

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Consider a Hausdorff topological space with Banach coördinates and with a symmetric linear connection $\Gamma(x, \xi_1, \xi_2)$. For postulates, definitions and notations, the reader is referred to the author's previous papers on General Geometry with a Linear Connection.[2] The defining equations of parallelism of a contravariant vector field $\xi(x)$ along a curve is given by

$$\frac{d\xi}{dt} + \Gamma\left(x, \xi, \frac{dx}{dt}\right) = \alpha(x)\xi(x), \tag{1}$$

where $\alpha(x)$ is a numerically valued scalar field. Hence the autoparallel curves (paths) satisfy a second order equation

$$\frac{d^2x}{dt^2} + \Gamma\left(x, \frac{dx}{dt}, \frac{dx}{dt}\right) = \alpha(x)\frac{dx}{dt}. \tag{2}$$

Eliminating $\alpha(x)$ in (1) we obtain

$$[\xi, \lambda]\left\{\frac{d\xi}{dt} + \Gamma\left(x, \xi, \frac{dx}{dt}\right)\right\} = \left[\frac{d\xi}{dt} + \Gamma\left(x, \xi, \frac{dx}{dt}\right), \lambda\right]\xi, \qquad (3)$$

where λ is an arbitrary covariant vector and $[x, y]$ is the independently postulated bilinear inner product—the Banach norm $||x||$ does *not necessarily* satisfy the relation $||x|| = [x, x]^{1/2}$ of a Hilbert space.[3] Similarly on eliminating $\alpha(x)$ in (2) we obtain the corresponding differential equation of the paths.

Let E be the Banach space with the independently postulated inner product $[x, y]$. The linear (additive and continuous) transformations in E form a new Banach space R whose norm function may be taken as the modulus of a linear transformation in E. In fact R is a *normed ring with unit* element. The bilinear ring product L_1L_2 is defined by the succession of two linear transformations in E and the unit I of R is defined by the identical transformation in E.

We condition the normed ring R, and hence indirectly the space E, by the following postulate.

There exists a linear function [L], *called contraction, on* R *to the real numbers with the properties* (1) $[L_1L_2] = [L_2L_1]$, (2) $[N(^*)\alpha] = N(\alpha)$, *where* $N(\xi)$ *is linear on* E *to the real numbers.*

It can be shown that *the most general transformation of the linear connection* $\Gamma(x, \xi_1, \xi_2)$, *called projective change of connection, that preserves the equation* (3) *of parallelism and hence the corresponding equations of paths, is given by*

$$'\Gamma(x, \xi_1, \xi_2) = \Gamma(x, \xi_1, \xi_2) + \phi(x, \xi_1)\xi_2 + \phi(x, \xi_2)\xi_1, \qquad (4)$$

where $\phi(x, \xi)$ *is any scalar field valued linear form in a contravariant vector* ξ.

Let $B(x, \xi_1, \xi_2, \xi_3)$ be the c.v.f. (contravariant vector field) valued *curvature form* vased on the linear connection $\Gamma(x, \xi_1, \xi_2)$ and define

$$\left.\begin{aligned} B(x, \xi_1, \xi_2) &= [B(x, \xi_1, {}^*, \xi_2)] \\ \beta(x, \xi_1, \xi_2) &= \tfrac{1}{2}[B(x, {}^*, \xi_1, \xi_2)] \\ \nu &= [I]. \end{aligned}\right\} \qquad (5)$$

It can be shown that $W(x, \xi_1, \xi_2, \xi_3)$ defined below is a c.v.f. valued trilinear form in the contravariant vectors ξ_1, ξ_2, ξ_3

$$\left.\begin{aligned} W(x, \xi_1, \xi_2, \xi_3) = B(x, \xi_1, \xi_2, \xi_3) &- \frac{2}{\nu + 1}\beta(x, \xi_2, \xi_3)\xi_1 - \\ \frac{1}{\nu - 1}&(B(x, \xi_1, \xi_3)\xi_2 - B(x, \xi_1, \xi_2)\xi_3) - \\ \frac{2}{\nu^2 - 1}&(\beta(x, \xi_1, \xi_2)\xi_3 - \beta(x, \xi_1, \xi_3)\xi_2). \end{aligned}\right\} \qquad (6)$$

It can also be shown that $W(x, \xi_1, \xi_2, \xi_3)$ is a *projective differential invariant*[3] i.e., it remains invariant under a projective change of connection.

A space with a linear connection will be said to be *projectively flat* if there exists a projective change of connection yielding a space with a vanishing curvature form. After several preliminary new theorems in the author's general tensor analysis, the following fundamental result in general projective geometry is proved with the aid of some additional restrictions including the Fréchet differentiability of the adjoint[4] $\Gamma^*_{(2)}(x, \eta, \xi)$. *If* $[I] \neq -1, +1, +2$, *then a necessary and sufficient condition that a space with a symmetric linear connection be projectively flat* (*locally*) *is that the projective curvature form* $W(x, \xi_1, \xi_2, \xi_3)$ *vanish.* In the proof of this theorem we make use of numerous recent results in the abstract differential calculus including an existence theorem for completely integrable differential equations in abstract spaces.[5]

In case E is specialized to be the function space of real continuous functions of a real variable the above results complete some tentative studies of the author in functional projective differential geometry.[6]

It is planned to give detailed developments elsewhere in a comprehensive paper.

[1] Presented to the American Mathematical Society, September 7, 1937.

[2] Michal, A. D., "Postulates for Linear Connections in Abstract Vector Spaces," *Annali di Matematica*, **15**, 197–220 (1936); "General Tensor Analysis," *Bull. Amer. Math. Soc.*, **43**, 394–401 (1937); "Abstract Covariant Vector Fields in a General Absolute Calculus," *Amer. Jour. Math.*, **59**, 306–314 (1937); "Riemannian Differential Geometry in Abstract Spaces," these PROCEEDINGS, **21**, 526–529 (1935).

[3] Since the Banach space is much more general than a Hilbert space, we restrict ourselves in this paper to those *numerically* valued linear functions $l(x)$ that can be written $l(x) = [\alpha, x]$.

[4] The adjoint of the linear connection considered as a linear function of the second variable.

[5] Michal, A. D., and Elconin, V., "Completely Integrable Differential Equations in Abstract Spaces," *Acta Mathematica*, **68**, 71–108 (1937).

[6] Michal, A. D., "Projective Functional Tensors and Other Allied Functionals," these PROCEEDINGS, **16**, 165–168 (1930).

GROUPS IN WHICH EVERY SUBGROUP OF COMPOSITE ORDER IS INVARIANT

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Since the non-abelian groups in which every subgroup is invariant have been determined we shall restrict ourselves here to a consideration of those groups which involve some non-invariant subgroup of prime order but in which every subgroup of composite order is invariant. If the order of such a group G is of the form p^m, p being a prime number, it contains an invariant subgroup of order p and the corresponding quotient group is abelian. Hence the commutator subgroup of G is of order p and this is the only invariant subgroup of order p contained in G since otherwise G would contain a non-invariant subgroup of order p^2 generated by a non-invariant subgroup of order p and an invariant subgroup of this order which is not the commutator subgroup. This proves the following theorem: *If a non-abelian group of order p^m, p being a prime number, contains no non-invariant subgroup of order p^2 then it contains only one invariant subgroup of order p.*

Every operator of order p^2 contained in G generates the commutator subgroup of order p since the subgroup of order p generated by such an operator is invariant under G. While G cannot contain any invariant operator of order p besides those in its commutator subgroup it may contain invariant operators of an arbitrary higher order. To construct a group containing such operators we may extend the non-abelian group of order p^3 which involves no operator of order p^2 when p is odd by a cyclic group of an arbitrary prime power order which involves the commutator subgroup of G and whose operators are commutative with every operator of the given group of order p^3. To construct an infinite system of groups in which no operators of the highest order are invariant we may start with the group of order p^2 and of type 1^2. This group may be extended by a cyclic group of arbitrary order which transforms this subgroup of order p^2 according to an operator of order p. The group thus obtained contains the direct product of a cyclic group of order p^{m-2} and a group of order p as a subgroup of index p and each of its remaining operators is of order p^{m-1} except when $p = 2$ and $m = 3$. In this special case it is the octic group.

When p is odd the operators of order p contained in G cannot generate a group whose order is as large as p^4 since otherwise G would contain a non-invariant subgroup of order p^2. Hence the operators of order p in such a G generate a subgroup whose order is either p^2 or p^3. In both cases the operators of order p^α, $\alpha > 0$, generate a subgroup of index p under the

group generated by the operators of order $p^{\alpha+1}$ if G contains operators of order $p^{\alpha+1}$. Hence the quotient group of G with respect to the subgroup generated by its operators of order p is cyclic in case G is not generated by its operators of order p. When the operators of order p contained in G generate a group of order p^3 this group is non-abelian since every subgroup of order p^2 contained in G is invariant. Hence it results that when the operators of order p contained in G generate a group of order p^3 then G involves invariant operators of highest order whenever $m > 3$ but when these operators generate a group of order p^2 no one of the operators of highest order contained in G is invariant under G.

It results from the preceding paragraph that when the operators of order p contained in G generate a subgroup of order p^2 then G is the well-known non-abelian group of order p^m which contains operators of order p^{m-1}, p being an odd prime number. The central of this group is cyclic and composed of the operators of G which are pth powers of other operators. When the operators of order p contained in G generate a group of order p^3 then G involves a cyclic subgroup of order p^{m-2} which constitutes its central. The direct product of this subgroup and a subgroup of order p is extended by an operator of order p which is non-commutative with some operators of order p contained therein to obtain G. Hence there results the following theorem: *When* p *is an odd prime there are two and only two groups of order* p^m, $m > 2$, *which have the property that every subgroup of composite order is invariant but that some subgroups of prime order are non-invariant.*

In the special case when $p = 2$ the commutator subgroup of G is still of order 2 and G involves one and only one invariant subgroup of order p but the operators of order p contained in G do not necessarily generate, in this case, a subgroup which involves no operator of order p^2. The fact that the group generated by these operators may be of order 32 is illustrated by the group obtained by extending the group of order 16 which involves 12 operators of order 4 having a common square by an operator of order 2 which transforms each of the operators of a quaternion subgroup into itself. To prove that the group generated by these operators could not be of order 64 it is only necessary to note that if it were of this order it would involve a non-invariant operator of order 2 which would be commutative with every operator of a subgroup of order 32 containing 28 operators of order 4 having a common square. Since this is impossible it has been proved that *if every subgroup of composite order in a group of order* 2^m *is invariant but some of the subgroups of order* 2 *are non-invariant then the subgroups of order* 2 *cannot generate a group whose order is as large as* 64.

From the preceding paragraph it results that the operators of order 2 contained in a non-abelian group of order 2^m whose subgroups of composite

order are separately invariant generate a group whose order is included in the following list: 4, 8, 16, 32. If such a group is of order 4 it is the non-cyclic group of this order. To prove that its order could not be 8 if the order of G exceeds this number it may first be noted that it could not then be the octic group since one of the remaining operators of order 4 and an operator of this order in the octic group would have a product of order 2. It could not be the abelian group of type 1^3 since every subgroup of order 4 contained in G is invariant under G. It could not be the group of order 16 which contains 12 operators having a common square, when the order of G exceeds 16, since the product of one of the operators of order 4 contained in this group and one of the remaining operators of order 4 would be of order 2. For the same reason it could not be the group of order 32 when its order exceeds this number.

If the group generated by the operators of order 2 in G is the four group, G involves a cyclic subgroup of index 2 since the quotient group with respect to the commutator subgroup is then of type 1, $m - 2$. The central of this group is cyclic and of order 2^{m-2} and there is one and only one such group of order 2^m whenever $m > 3$. When the group generated by its operators of order 2 is obtained by extending the octic group by an operator of order 4 which is commutative with each of its operators and has its square in this octic group then there is again one and only one such G of order 2^m, $m > 3$. This is obtained, when $m > 4$, by extending successively the given group of order 16 by an operator of order 8, an operator of order 16, etc., which is commutative with each of its operators. Hence there results the following theorem: *There are two and only two groups of order* 2^m, $m > 5$, *which have the property that every subgroup of order* 4 *contained therein is invariant but some subgroups of order* 2 *are non-invariant.*

No one of these groups is generated by its operators of order 2. The octic group is the only group of order 8 which is generated by its operators of order 2 and has the property that each of its subgroups of order 4 is invariant but that it contains non-invariant subgroups of order 2. There are two such groups of order 16. One of these is characterized by the fact that it contains twelve operators of order 4 which have a common square while the other belongs to one of the systems noted in the theorem at the end of the preceding paragraph. It was noted above that there is a group of order 32 which is generated by its operators of order 2 and has the property that each of its subgroups of composite order is invariant but that it contains non-variant operators of order 2 and that it is not a subgroup of a larger group having these properties. In particular, the direct product of this group and a group of order 2 does not have these properties. In fact, if a group is the direct product of a non-abelian group whose subgroups of composite orders are invariant

and a group of type 1^n it cannot have the property that each of its subgroups of composite order is invariant.

If the order of G is not a power of a prime number each of its Sylow subgroups whose order exceeds a prime number is invariant under G and hence G involves invariantly the direct product of all its Sylow subgroups of composite orders. The quotient group of G with respect to this direct product has an order which is the product of distinct prime numbers. In particular, it results that *whenever every subgroup of composite order contained in a group is invariant then the group is solvable.* If the given quotient group involves a non-invariant operator of prime order which is not equal to the smallest prime number which divides the order of this quotient group then this operator is non-commutative with an operator of smallest order in this quotient group and this pair of orders is the only pair of prime divisors of the order of this quotient group such that these divisors are the orders of two non-commutative operators of this quotient group. This fact results directly from the condition that every subgroup of composite order contained in G is invariant under G. Hence it results that when this quotient group is non-cyclic it is the direct product of a cyclic group and a group whose order is the product of two prime numbers, one of which is the smallest prime number which divides the order of this quotient group.

ON THE CONCEPT OF CO-SETS IN A SEMI-GROUP

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As in other papers[1] we define a semi-group as a discrete set of elements closed under an associative operation and for which the usual equality axioms hold.

If C is an element of a semi-group S such that from any relation of the type $CA = CB$, with A and B in S we may infer $A = B$ then C is said to be a left cancellable element in S, and if the order is reversed, a right cancellable element in S. Elements in S not having one of these properties are called left or right non-cancellable elements, respectively.

We define a *left-identity* in S as an element E_l in S such that $E_lM = M$ for any M in S with an analogous definition of right-identity. If an element is both a right and left identity we call it an identity element. If an identity of S exists, it is unique. An element O_r such that $MO_r = O_r$ for each M in S will be called a right annulator in S and if the

order is reversed we call the corresponding O_l a left annulator. If an element is both a right and left annulator, we call it an annulator. If an annulator exists in S it is unique. If S is a semi-group we shall term a subset of S a sub-semi-group of S if it satisfies the postulates for a semi-group.

In the present paper we shall state some theorems, concerning sub-semi-groups of S which are also groups, and develop the concept of co-sets in a semi-group with respect to an included group. This leads to generalizations of other concepts in group-theory such as invariant subgroup and conjugate sets. We also set up the concept of repetitive sets in a semi-group and show its relation to group theory and parts of the theory of numbers. Proofs of the results will be published elsewhere.

Let a semi-group S contain a sub-semi-group S' and let the elements of the latter be

$$N_1, N_2, N_3, \ldots . \tag{1}$$

Let C be an element of S and suppose that

$$A_1, A_2, A_3, \ldots . \tag{2}$$

are the distinct elements of (1) such that

$$C = CA_1 = CA_2 = .$$

The elements (2) form a semi-group which we shall call the *right semi-group in S belonging to C with respect to S'*, with an analogous definition for left semi-group belonging to C with respect to S'. We term the number of distinct elements in (2) the *right-spread of C in S with respect to S'* with an analogous definition of left-spread. We then have the

THEOREM 1. *Let a given semi-group S contain a group G consisting of the elements* (1) *and let C be any element of S. Further, suppose the identity element of G is a right (left)-identity of S. Let G' be the right (left) sub-semi-group (necessarily a group) belonging to C with respect to G. Assume that the elements of G can be distributed into right (left) co-sets with respect to G'. Then the distinct elements in the set*

$$CN_1, CN_2, CN_3, \ldots \tag{3}$$

are the elements

$$CB_1, CB_2, CB_3, \ldots \tag{4}$$

where $B_1, B_2, B_3 \ldots$ are the multipliers of the right (left) co-sets of G with respect to G'.

We are employing the constructive point of view, so we insert specifically in the above statement the assumption that the elements of the group G may be distributed into co-sets with respect to the included group G which we regard as a necessary assumption unless G is finite.

In view of the above theorem we may set up a group which depends, curiously, on the notion of non-cancellable elements in a semi-group. For, employing the notation and assumptions of Theorem 1 let the elements (3) be connected by an operation, designated by $\circ$ and defined by

$$CN_i \circ CN_j = CN_iN_j.$$

Then it may be shown that the distinct elements (4) form a group under this type of operation.

We now introduce the notion of co-sets of a semi-group S with respect to an included group G. Write $C = C_1$. If there are no other elements in S than those in (4) then G corresponds to S. If there is another element in S, say C_2, then consider

$$C_2N_1,\ C_2N_2,\ C_2N_3,\ \ldots. \tag{5}$$

The distinct elements of this set are obtained from Theorem 1, and each may be shown to be different from the elements (4). The sets (4) and (5) are said to be left co-sets of S with respect to G. Proceeding further in this manner, if necessary, we may possibly represent the elements of S in a series of co-sets. If S is finite this process may be carried out in a finite number of steps and we have the

THEOREM 2. *Let a finite semi-group S contain a group G and assume that the identity element of G is a right (left)-identity of S. Then the distinct elements of S may be distributed into left (right) co-sets with respect to G. If n is the order of G then $s_it_i = n$ where s_i is the right (left) spread of C_i (the multiplier of the i-th co-set) with respect to G. If g is the order of S then*

$$g = \sum_{i=1}^{h} \frac{n}{s_i}$$

where h is the number of co-sets. For a given S and G, each s is an invariant.

As an application of this theorem to a special case let S be the semi-group of residue classes modulo 24 under multiplication and let G be the group of residue classes modulo 24 corresponding to integers prime to 24. Then we may set up a system of eight co-sets for S with respect to G and the spreads of the multipliers are 1, 2, 2, 4, 4, 4, 8, 8.

Let S be a commutative (Abelian) semi-group and S' a sub-semi-group of S. Consider a finite sub-set R of S say

$$C_1,\ C_2,\ C_3,\ \ldots \tag{6}$$

and an element M of S such that

$$MC_1,\ MC_2,\ \ldots\ MC_k, \tag{7}$$

gives a permutation of the elements

$$a_1C_1,\ a_2C_2,\ \ldots\ a_kC_k, \tag{8}$$

where the a's belong to S'. Then R is said to be a *repetitive set in S, with respect to S', and with multiplier M.* If S contains an identity E, then if we set $a_1 = a_2 = \ldots = a_k = E$ we obtain a set which we call simply a repetitive set in S with multiplier M. If S contains no identity we give a special definition to cover the case where no a's appear in (6).

The multipliers C_i of the co-sets in S (assumed commutative) mentioned in Theorem 2 may be shown to form a repetitive set with respect to G and with any element of S as multiplier. This idea is the abstract form of many relations in number theory. With the notation and definitions of (6), (7) and (8) we have easily, if the C's are cancellable elements of S,

$$M^k = \prod_{s=1}^{k} a_s. \tag{9}$$

In particular this result may be applied to the theory of algebraic fields to obtain known theorems concerning quadratic and higher residues of ideal moduli. If S is the semi-group of ideal classes moduli p, a rational prime, consisting of the elements

$$C_1, C_2, \ldots, C_{p-1}$$

and G is the included group consisting of C_1 and C_{p-1}, then (9) gives Gauss' lemma in the theory of quadratic residues.

[1] These PROCEEDINGS, **20**, 579 (1934); *Bull. Am. Math. Soc.*, **40**, 916 (1934).

ON GENERALIZATIONS OF THE NUMBERS OF BERNOULLI AND EULER

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In the present paper we shall develop a number of generalizations of the Bernoulli and Euler numbers and results concerning the same. Full proofs of the results will appear elsewhere.

Our point of departure is the examination of the expression

$$(mb + k)^n = b_n(m, k). \tag{1}$$

Where m and k are integers, $m \neq 0$ and we expand the left-hand member by the binomial theorem and substitute b_i for b^i in the result, where b_i is defined by the recursion formula

$$(b + 1)^a = b_a; \quad a > 1; \quad b_0 = 1. \tag{2}$$

The numbers so obtained from (1) have arithmetical properties analogous to those known for the ordinary Bernoulli numbers (2) and which are little more complicated. For example, we have the following generalization of the von-Staudt-Clausen theorem:

THEOREM 1. *We have for n even, m and k integers with* $m \neq 0$,

$$b_n(m, k) = A_n - \sum_{i=1}^{s} \frac{1}{p_i}, \tag{3}$$

where the p's are the distinct primes which are prime to m and such that $n \equiv 0 \pmod{p_i - 1}$; A_n *being some integer. For n odd,* $b_n(m, k)$ *is an integer, except for* $n = 1$ *with m odd.*

Note the peculiarity that this result is independent of k and that it reduces to the ordinary von-Staudt-Clausen theorem for $m = 1$, $k = 0$. It may be proved by taking the known relation

$$\frac{(mb + k + mp)^{n+1} - (mb + k)^{n+1}}{(n + 1)m} = \sum_{s=0}^{p-1} (ms + k)^n \tag{4}$$

for p prime, with $0^0 = 1$, expanding the left-hand member in ascending powers of p, and then extending and modifying the argument employed by von-Staudt and Frobenius[1] for proving the von-Staudt-Clausen theorem.

An advantage from some standpoints in using (1) instead of (2) is the fact that $b_n(m, k)$ is not always zero for n odd.

We shall call $b_n(m, k)$; $m \neq 0$, *a generalized Bernoulli number of the first order:* and a number of the form, for $r \geq 1$,

$$(m_r b^{(r)} + m_{r-1} b^{(r-1)} + \ldots + m_1 b' + m_0)^n = b_n(m_r, m_{r-1}, \ldots, m_0), \tag{5}$$

where this expression is expanded in full by the multinomial theorem and b_a substituted for $b_a^{(i)}$; $i = 1, 2, \ldots r$ and where the m's are integers, $m_i \neq 0$; $i = 1, 2, \ldots r$; *a Bernoulli number of the r-th order.* This is an extension of the definition of Lucas of Ultra-Bernoulli numbers.

It is now natural to inquire if a generalization of the von-Staudt-Clausen theorem exists for numbers of the type (5). For the case $m_0 = 0$, $r = 2$ we may employ formula (3) of another paper,[2] treat it as in the derivation of (6) of that paper except that we differentiate a times, a arbitrary, set $v = 0$, then substitute $x = xe^{vk}$ but not as before $y = ye^{vj}$ so that we get rid of $(x^j - y^k)$ as a factor of the left-hand member when we differentiate once more. Then set $v = 0$, take the result modulo p^3 and set $x = y = 1$ in the resulting congruence, and divide through by p^2, using the relation, (which may be derived from (4))

$$\sum_{s=0}^{p-1} (ms + k)^n \equiv p(mb + k)^n \pmod{p^2};\ p > n, \tag{6}$$

where $0^0 = 1$. Our result, which is an equality as well as a congruence since the terms are independent of p, expresses

$$(kb' + jb'')^a \tag{7}$$

as a linear combination of expressions of the form $(kb + l)^n$ and $(jb + i)^n$; $n = a, a - 1$. In view of Theorem 1 we are able to determine directly the fractional part of (7) and this is analogous to the result of Theorem 1.

I conjecture that a similar result holds for (5), but the question is much more complicated for higher values of r.

The method just employed is quite general in connection with deriving formulas concerning Bernoulli numbers. The simplicity of the formula (6), especially for $k = 0, m = 1$, and the fact that $(mb + k)^n$ is independent of p, enables us to obtain many equalities involving the generalized Bernoulli numbers merely by taking simple identities containing expressions such as

$$\frac{x^{pm} - 1}{x - 1}$$

where x is an indeterminate, setting $x = e^{av}x$ and differentiating with respect to v, finally reducing the result to a congruence modulo p, which usually gives an equality because the terms therein are independent of p. As another example, from formula (11) of my paper already referred to we may obtain a formula closely related to (12) of that paper, which gives, after substituting $x = y = 1$, congruences from which we may eventually derive equations involving $b_n(2, 1)$ and more generally $b_n(m, k)$ as indicated by the remarks below (12).

Now the generalization of the Bernoulli numbers as given by (1) suggests analogous extensions for certain types of related numbers. We note that

$$(mb + k)^n = m^n\left(b + \frac{k}{m}\right)^n \equiv m^n(b + s)^n (\text{mod } d)$$

where d is an integer prime to m and k, and s is the least positive solution of $sm \equiv k (\text{mod } d^2)$. Using (4) for $m = 1, k = 0$ we obtain

$$(mb + k)^n - m^n b_n \equiv nm^n S_{n-1}\left(\frac{k, d^2}{m}\right) (\text{mod } d)$$

where

$$S_{n-1}\left(\frac{k, d^2}{m}\right) = 1^{n-1} + 2^{n-1} + \ldots + (s - 1)^{n-1}.$$

From this we have

$$(mb + k)^n - (mb + l)^n \equiv nm^n S_{n-1}\left(\frac{k, d^2}{m}\right) - nm^n S_{n-1}\left(\frac{l, d^2}{m}\right)$$

modulo d. Let $d = nd_1$, then if we set

$$e_{n-1}(m, k, l) = \frac{(mb + k)^n - (mb + l)^n}{n} \tag{8}$$

we have

$$e_{n-1}(m, k, l) \equiv m^n S_{n-1}\left(\frac{k, d^2}{n}\right) - m^n S_{n-1}\left(\frac{l, d^2}{m}\right) \tag{9}$$

modulo d_1, where m, k and l are prime to d_1. We shall call $e_n(m, k, l)$ if m, k and l are integers with $m \neq 0$, $k \neq l$; a *generalized Euler number* but shall not at this time refer to its order, as different generalizations[3] of the Euler numbers have been proposed. The relation of (8) to the ordinary Euler numbers is given by

$$e_n(4, 3, 1) = 2e_n,$$

where

$$(-e^2)^n = E_n.$$

Our e's in (8) are not always integers but the prime factors of the denominators always divide m. However, it appears that most of the known arithmetical properties of the E's have analogons involving the e's, and a number of them may be obtained from (9). For example, the theorem I gave in another paper[4] may be applied to (9) to obtain results such as that given at the top of page 422 of that paper with $e_n(m, k)$ taking the place of b_n/n as there given.

An analogous extension of the Genocchi numbers is given by

$$G_n(m, k) = -((mb + k)^n - b_n)m, \tag{10}$$

which reduces to the ordinary Genocchi numbers for $k = 0$, $m = 2$.

There is another type of number, however, which has appeared repeatedly in investigations concerning Bernoulli numbers, and which deserves special consideration since its properties differ in many respects from those of the Bernoulli number itself. It is

$$\frac{(m^n - 1)b_n}{n} \tag{11}$$

of which a natural generalization appears to be

$$\frac{(mb + k)^n - b_n}{n} = t_n(m, k). \tag{12}$$

For $m = 2$, (11) is closely related to the so-called tangent coefficient

$$\frac{2^n(2^n - 1)}{n} b_n, \tag{13}$$

that is, the coefficient of $x^{n-1}/(n-1)!$ in the power series expansion of $\tan x$. As is also known, the number (11) appears explicitly as the coefficient of $x^{n-1}/(n-1)!$ in the expansion of

$$\frac{d}{dx} \log \frac{e^{mx}-1}{e^x-1}$$

as a power series in x.

Although (12) is not always an integer we prefer its use to that of

$$m^n \frac{(mb+k)^n - b_n}{n},$$

which is an integer divisible, in general, by a power of m close to m^n. Concerning (12) we have found a reduced form for

$$(t'(m_1, k_1) + t^{(2)}(m_2, k_2) + \ldots + t^{(r)}(m_r, k_r))^n,$$

where this means that the expression is to be expanded by the multinomial theorem and $t_s(m_i, k_i)$ substituted for $(t^{(i)}(m_i, k_i))^s$; $i = 1, 2, \ldots, r$; which is much simpler than those found for (5).

[1] *Sitsungsberichte Berlin Akad.*, 816–817 (1910).
[2] *Ann. Math.*, **29**, 451–458 (1928).
[3] Sylvester, *Compt. Rend.*, **52**, 163; D. H. Lehmer, *Ann. Math.*, **36**, 648–649 (1935).
[4] *Bull. Amer. Math. Soc.*, **43**, 420–421 (1937).